Rationals Whose Sum Equals the Reciprocal of their Product

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For which positive integers n do there exist positive integers $a_1, \ldots, a_{n+1}$ such that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_n}{a_{n+1}} = \frac{a_{n+1}}{a_1}\,? \tag{1}$$

The second author proposed a special case of this problem for the 2002 IMO, but it was not shortlisted. The full problem was then given to the first author to work on, as part of his 2003 Summer NSERC Undergraduate Research Fellowship project. This note is the result.

We first restate the problem in a more convenient form.

Problem: For which positive integers n do there exist positive rationals $b_1, \ldots, b_n$ such that

$$b_1 + b_2 + \cdots + b_n = \frac{1}{b_1 b_2 \cdots b_n}\,? \tag{2}$$

Given a positive integer solution to (1), we obtain a positive rational solution to (2) simply by letting $b_i = a_i/a_{i+1}$ for each integer i, $1 \le i \le n$. Conversely, given a positive rational solution to (2), we can obtain a positive rational solution to (1) by choosing an arbitrary positive rational a_1 and letting $a_{i+1} = a_i/b_i$ for each integer i, $1 \le i \le n$. From there we can obtain a positive integer solution by multiplying all of the a_i's by a suitable integer. Note that (2) is symmetric in all of the b_i's, and that reordering the b_i's can give distinct solutions to (1).

The case $n = 1$ has the obvious solution $b_1 = 1$. For $n = 2$, it is an old, but not widely known, fact that there is no solution to this problem [2]; that is, *there is no solution in positive integers to*

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} = \frac{a_3}{a_1}\,.$$

We next include a proof for the convenience of the reader.

We rewrite the above equation by putting $A = a_1$, $B = a_2$, $C = a_3$, and clearing denominators, obtaining

$$A^2C + B^2A = C^2B\,. \tag{3}$$

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(This is the form appearing in [2].) Suppose (3) has a solution in positive integers. We can clearly assume that A, B, C have no common factor greater than 1.

Let p be a prime dividing both A and B (but not C). Write $A = p^a A'$ and $B = p^b B'$, where a, b, A', $B' \geq 1$ and $(p, A') = (p, B') = 1$. Then (3) becomes

$$p^{2a} A'^2 C + p^{2b+a} A' B'^2 \;=\; p^b B' C^2 \,,$$

where $(p, C) = 1$. Note that if $2b + a \leq 2a$, then p^{2b+a} must divide the right side of this equation, which forces $2b + a \leq b$, an impossibility. Therefore, $2a < 2b + a$, which forces both sides of the equation to be divisible by p^{2a} and by no larger power of p. Thus, $2a = b$.

This holds for every prime p dividing both A and B. Hence, letting $R = (A, B)$, we can write $A = RS$ and $B = R^2 T$ for positive integers R, S, T satisfying $(R, C) = (R, S) = (R, T) = (S, T) = 1$. Equation (3) now becomes $R^2 S^2 C + R^5 S T^2 = R^2 T C^2$, which simplifies to

$$S^2 C + R^3 S T^2 \;=\; T C^2 \,. \tag{4}$$

Since $(S, T) = 1$, we get $T \mid C$. Then TC^2 and $R^3 S T^2$ are both divisible by T^2, forcing $T^2 \mid C$. Writing $C = T^2 D$ and cancelling turns (4) into

$$S^2 D + R^3 S \;=\; T^3 D^2 \,. \tag{5}$$

Since $(D, R) = 1$, we get $D \mid S$, which (as above) forces $D^2 \mid S$. But since $(S, T) = 1$, we also get from (5) that $S \mid D^2$, which means that $S = D^2$. Thus, (5) simplifies to

$$D^3 + R^3 \;=\; T^3 \,.$$

This is the exponent 3 case of Fermat's Last Theorem and is known to have no solution in positive integers (see for instance [3], [4]). □

For $n > 2$, the only reference we have found is [1], where solutions due to Euler are given for the values $n = 3$ and $n = 4$. However, the solution for $n = 4$ contains negative rationals, and Euler's purpose apparently was not to consider equation (2) in general.

Anyway, here is our result.

Theorem. The equation

$$b_1 + b_2 + \cdots + b_n \;=\; \frac{1}{b_1 b_2 \cdots b_n}$$

has a positive rational solution for each positive integer $n \geq 3$.

Proof: We consider four cases, depending on the congruence class of n modulo 4, and exhibit solutions in each case. Here and below, let b denote $(b_1, b_2, \ldots, b_n)$, although, as we mentioned above, the order of the b_i's is not important.

The following table contains a solution for each of the various cases, where, for example, we write $(1/k)^k$ to denote k copies of $1/k$.

Case	Solution $b_1, b_2, \ldots, b_n$	$\sum_{i=1}^{n} b_i$
$n \equiv 1 \pmod 4$		
$n = 4k-3,\ k \geq 2$	$\left(\frac{1}{k}\right)^k,\ k^{k-2},\ 1^{2k-1}$	k^2
$n \equiv 3 \pmod 4$		
$n = 3$	$\frac{4}{3},\ \frac{3}{2},\ \frac{1}{6}$	3
$n = 4k-5,\ k \geq 3$	$\left(\frac{1}{k}\right)^k,\ k^{k-2},\ 1^{2k-6},\ \frac{1}{2},\ \frac{1}{2},\ 4$	k^2
$n \equiv 0 \pmod 4$		
$n = 4$	$\frac{1}{6},\ \frac{1}{3},\ 2,\ 2$	$\frac{9}{2}$
$n = 4k-4,\ k \geq 3$	$\left(\frac{1}{k}\right)^k,\ k^{k-2},\ 1^{2k-6},\ \frac{1}{2},\ \frac{1}{2},\ 2,\ 2$	k^2
$n \equiv 2 \pmod 4$		
$n = 6$	$\frac{1}{6},\ \frac{1}{3},\ 1,\ 1,\ \frac{3}{2},\ 2$	6
$n = 10$	$\frac{1}{18},\ \frac{1}{9},\ \frac{1}{3},\ \frac{1}{2},\ 1,\ 1,\ 1,\ 2,\ 3,\ 9$	18
$n = 14$	$\frac{1}{6},\ \left(\frac{1}{3}\right)^4,\ \left(\frac{1}{2}\right)^3,\ 2^3,\ 3^3$	18
$n = 4k-6,\ k \geq 6$	$\left(\frac{1}{k}\right)^k,\ k^{k-2},\ 1^{2k-12},\ \left(\frac{1}{3}\right)^3,\ 3,\ \left(\frac{3}{2}\right)^2,\ 2^2$	k^2

For example, when $n = 5$, we put $k = 2$ to get the solution $\mathrm{b} = (1/2, 1/2, 1, 1, 1)$, yielding the following solution to (1):

$$\frac{1}{2} + \frac{2}{4} + \frac{4}{4} + \frac{4}{4} + \frac{4}{4} = \frac{4}{1}.$$

By reordering b, we could get other solutions to (1) differing slightly from this one. □

We remark that the number of distinct positive rational solutions to (2) grows arbitrarily large as n increases. (Two solutions that are permutations of each other are not considered distinct.) In particular,

(2) *has at least* $\left\lceil \frac{n-22}{36} \right\rceil$ *distinct positive rational solutions for any* n.

Fix n. Let $d_i = \lceil (5i-1)/2 \rceil \geq 0$ for $0 \leq i \leq 3$, and write

$$n = 4(9t + r + d_s) - s + 1$$

for integers r, s, t, where $0 \leq r \leq 8$, $0 \leq s \leq 3$; this is always possible uniquely, since s is determined by the congruence class of n modulo 4, and then $(n+s-1)/4 - d_s$ can be written uniquely as $9t + r$. If n is sufficiently large (precisely, $n > 22$), then $t \geq 0$.

Suppose that $n > 22$. We claim that (2) has the $t+1$ distinct solutions given by

$$\mathbf{b} \;=\; \left(\left(\frac{1}{k}\right)^{k}, k^{k-2}, 1^{2k-1-20i-5s}, \left(\frac{1}{2}\right)^{8i+2s}, 2^{8i+2s}\right) \tag{6}$$

as i ranges over the integers from 0 to t, where $k = 9t + d_s + r + i + 1 \geq 1$. Note that

$$2k-1-20i-5s \;=\; 18(t-i)+(2d_s-5s)+2r+1 \;\geq\; 0-1+0+1 \;=\; 0\,,$$

and the length of $\mathbf{b}$ comes out to be

$$4k-3-4i-s \;=\; 36t+4d_s+4r+1-s \;=\; n\,.$$

Therefore, the vector in (6) does indeed have length n. It is easily verified that $\sum\limits_{j=1}^{n} b_j \;=\; k^2$ and $\prod\limits_{j=1}^{n} b_j \;=\; \frac{1}{k^2}$. Therefore, (2) holds for each of the solutions in (6). Finally, the number of solutions is

$$\begin{aligned} t+1 \;&=\; \frac{n+35-4r-(4d_s-s)}{36} \\ &\geq\; \left\lceil \frac{n+35-32-(4d_3-3)}{36} \right\rceil \;=\; \left\lceil \frac{n-22}{36} \right\rceil \,. \end{aligned}$$

To end, here is an unsolved problem, suggested by Filip Saidak:

For which positive integers n, if any, does equation (2) *have infinitely many positive rational solutions?*

References.

[1] L. E. Dickson, *History of the Theory of Numbers, Volume II*, Chelsea, 1992, p. 648 (item 187).

[2] H. S. Vandiver, W. F. King, Solution to Problem 101, *Amer. Math. Monthly* **10** (1903), p. 22.

[3] P. Ribenboim, *13 Lectures on Fermat's Last Theorem*, Springer-Verlag, 1979, pp. 39–45.

[4] R. D. Carmichael, *Diophantine Analysis*, Wiley, 1915.

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