

# How the Roots of a Polynomial Vary with its Coefficients: A Local Quantitative Result

Bernard Beauzamy

*Abstract.* A well-known result, due to Ostrowski, states that if  $\|P - Q\|_2 < \varepsilon$ , then the roots  $(x_j)$  of  $P$  and  $(y_j)$  of  $Q$  satisfy  $|x_j - y_j| \leq Cn\varepsilon^{1/n}$ , where  $n$  is the degree of  $P$  and  $Q$ . Though there are cases where this estimate is sharp, it can still be made more precise in general, in two ways: first by using Bombieri's norm instead of the classical  $l_1$  or  $l_2$  norms, and second by taking into account the multiplicity of each root. For instance, if  $x$  is a simple root of  $P$ , we show that  $|x - y| < C\varepsilon$  instead of  $\varepsilon^{1/n}$ . The proof uses the properties of Bombieri's scalar product and Walsh Contraction Principle.

## 1 The General Theory

A well-known result due to Ostrowski [6], [7] can be stated as follows:

(1) Let  $P = \sum_0^n a_{n-j}z^j$ ,  $Q = \sum_0^n b_{n-j}z^j$ , be two polynomials, satisfying  $a_0 = b_0 = 1$ , and with respective roots  $x_1, \dots, x_n, y_1, \dots, y_n$ . Let

$$T = \max\{1, |a_1|, |b_1|, \dots, |a_k|^{1/k}, |b_k|^{1/k}, \dots, |a_n|^{1/n}, |b_n|^{1/n}\}.$$

Then, if the  $y_j$ 's are suitably ordered, we have, for all  $j$ ,

$$|x_j - y_j| \leq 4nT\delta^{1/n}$$

with

$$\delta = \left( \sum_0^n |a_j - b_j|^2 \right)^{1/2}.$$

(2) Let  $P, Q$  be as before; assume moreover that 0 is not a root of  $P$ . Assume that, for all  $j$

$$|a_j - b_j| \leq \tau |a_j|$$

where  $\tau$  is small enough, namely

$$\tau \leq \left( \frac{1}{4n} \right)^n.$$

Then the zeros  $y_j$ 's of  $Q$  can be ordered in such a way that

$$\left| \frac{y_j}{x_j} - 1 \right| < 8n\tau^{1/n}, \quad \text{for all } j.$$

---

Received by the editors July 27, 1996; revised April 22, 1998.  
 In application of contract ETCA/CREA/20461/95, Ministry of Defense, France.  
 AMS subject classification: 30C10.  
 ©Canadian Mathematical Society 1999.

Here, we will make this result more precise, in two ways. First, in order to measure  $P - Q$ , we will use Bombieri's norm, and second, we will take into account the multiplicity of the roots, which, of course, may be different from one to the other: this is why we speak of a "local" result.

Let  $P = \sum_0^n a_j z^j$  be a polynomial with complex coefficients and degree  $n$ . Its Bombieri's norm is defined by

$$(1) \quad [P] = \left( \sum_0^n \frac{|a_j|^2}{\binom{n}{j}} \right)^{1/2}.$$

This definition is better understood in its original frame, that of homogeneous many-variable polynomials: see Beauzamy-Bombieri-Enflo-Montgomery [3] and Beauzamy-Dégot [4].

Let  $x_1, \dots, x_n$  be the roots of  $P$ .

Let  $Q$  be another polynomial, with same degree, satisfying

$$(2) \quad [P - Q] \leq \varepsilon.$$

**Theorem 1** *If  $x$  is any zero of  $P$ , there exists a zero  $y$  of  $Q$ , with*

$$(3) \quad |x - y| \leq \frac{n(1 + |x|^2)^{n/2}}{|Q'(x)|} \varepsilon.$$

*If  $\varepsilon$  is small enough, namely*

$$(4) \quad \varepsilon \leq \frac{1}{2} \frac{|P'(x)|}{n(1 + |x|^2)^{\frac{n-1}{2}}}$$

*then (3) implies*

$$(5) \quad |x - y| \leq \frac{2n(1 + |x|^2)^{n/2}}{|P'(x)|} \varepsilon.$$

Before we turn to the proof, let us make some comments about these results.

– Estimates (3) and (5) are homogeneous (which is already an improvement upon Ostrowski's result). Indeed, if all coefficients of  $P$  and  $Q$  are multiplied by  $\lambda$ , so is  $\varepsilon$ , and  $\varepsilon/|Q'(x)|$  or  $\varepsilon/|P'(x)|$  are not modified.

– Theorem 1 is empty if  $x$  is not a simple root, either for  $P$  or for  $Q$  (note that  $Q$  can have all roots simple, and  $P$  have only one root, as the example of  $z^n$  and  $z^n + \alpha$  shows).

– The term  $(1 + |x|^2)^{1/2}$  can itself be bound by a quantity depending only on the coefficients of the polynomial, for instance by  $(1 + R^2)^{1/2}$ , where  $R$  is the radius of the largest disk, centered at 0, containing all the zeros. An estimate for  $R$  can be found in Marden [5]; others may be given, using for instance Mahler's measure of  $P$ . Here, we will give later (Theorem 4) a bound depending on Bombieri's norm  $[P]$ .

**Proof of Theorem 1** We need a few simple facts about Bombieri's norm, and the corresponding scalar product, which is just

$$(6) \quad [P, Q] = \sum_{j=0}^n \frac{a_j \bar{b}_j}{\binom{n}{j}},$$

if  $P = \sum_{j=0}^n a_j z^j$ ,  $Q = \sum_{j=0}^n b_j z^j$ .

**Lemma 2 (B. Reznick [8])** For any  $z_0$ ,

$$P(z_0) = [P, (\bar{z}_0 z + 1)^n].$$

(See Reznick [8] or Beauzamy-Dégot [4] for a proof.)

As a consequence, we get

$$(7) \quad |P(z_0)| \leq [P](1 + |z_0|^2)^{n/2}.$$

Indeed

$$|P(z_0)| = |[P, (\bar{z}_0 z + 1)^n]| \leq [P][(\bar{z}_0 z + 1)^n],$$

and an immediate computation shows that

$$[(\alpha z + 1)^n] = (1 + |\alpha|^2)^{n/2}.$$

Another property of the scalar product is

$$(8) \quad [P', R] = n[P, zR]$$

if  $\deg P = n$ ,  $\deg R = n - 1$  (see [4] for a proof).

**Lemma 3** If  $f(z) = az + b$  ( $a \neq 0$ ) satisfies  $|f(z_0)| \leq \varepsilon$ , there exists  $z_1$ , with  $|z_1 - z_0| \leq \varepsilon/|a|$ , such that  $f(z_1) = 0$ . More generally, if  $f(z) = a(z - z_1) \cdots (z - z_k)$  satisfies  $|f(z_0)| \leq \varepsilon$ , one of the roots, say  $z_1$ , satisfies

$$|z_1 - z_0| \leq (\varepsilon/|a|)^{1/k}.$$

**Proof of Lemma 3** This is obvious: if

$$|z_0 - z_1| \cdots |z_0 - z_k| \leq \frac{\varepsilon}{|a|},$$

one of the  $|z_0 - z_j|$  must be at most equal to  $(\varepsilon/|a|)^{1/k}$ .

Let us now prove the theorem. Since  $x$  is a root of  $P$ , we have, by (7):

$$(9) \quad |Q(x)| = |(Q - P)(x)| \leq \varepsilon(1 + |x|^2)^{n/2}.$$

Set

$$\varepsilon' = \varepsilon(1 + |x|^2)^{n/2}.$$

We know by Lemma 2 that

$$(10) \quad Q(x) = [Q, (\bar{x}z + 1)^n].$$

Let us consider

$$f(\zeta) = [Q, (\bar{x}z + 1)^{n-1}(\bar{\zeta}z + 1)],$$

which is an affine function of  $\zeta$ , satisfying

$$(11) \quad |f(x)| \leq \varepsilon'.$$

By Lemma 3, there is a point  $x'$ ,  $|x' - x| \leq \varepsilon'/|a|$  (where  $a$  is the coefficient of  $\zeta$  in  $f$ ), such that  $f(x') = 0$ . Let's compute  $a$ . By definition:

$$\begin{aligned} a &= [Q, (\bar{x}z + 1)^{n-1}z] \\ &= \frac{1}{n} [Q', (\bar{x}z + 1)^{n-1}] \quad \text{by (8)} \\ &= \frac{1}{n} Q'(x), \end{aligned}$$

by Lemma 2. So we see that a zero  $x'$  of  $f$  satisfies

$$(12) \quad |x' - x| \leq \frac{n\varepsilon'}{|Q'(x)|}.$$

Let us now apply Walsh Contraction Principle (Walsh [9], see Beauzamy [1] for a detailed study and proof). Consider

$$(13) \quad \varphi(u_1, \dots, u_n) = [Q, (\bar{u}_1z + 1) \cdots (\bar{u}_nz + 1)].$$

This is a symmetric function of  $u_1, \dots, u_n$ , affine with respect to each of them. It satisfies  $\varphi(x, \dots, x, x') = 0$ . Therefore, in each disk containing both  $x$  and  $x'$ , and in particular in the disk of diameter  $xx'$ , there is a point  $y$  such that

$$(14) \quad \varphi(y, \dots, y) = 0.$$

Coming back to the definition of  $\varphi$ , we get

$$\varphi(y, \dots, y) = [Q, (\bar{y}z + 1)^n] = Q(y).$$

So  $y$  is a zero of  $Q$ . Since it is in the disk of diameter  $xx'$ , we have also by (12):

$$|x - y| \leq \frac{n\varepsilon'}{|Q'(x)|},$$

and the first part of Theorem 1 is proved. To get the second part, we write simply:

$$\begin{aligned} |P'(x) - Q'(x)| &= |[P' - Q', (\bar{x}z + 1)^{n-1}]| \\ &= n|[P - Q, z(\bar{x}z + 1)^{n-1}]| \\ &\leq n[P - Q][z(\bar{x}z + 1)^{n-1}] \\ &\leq n\varepsilon(1 + |x|^2)^{\frac{n-1}{2}}. \end{aligned}$$

So  $|Q'(x)| \geq |P'(x)| - n\varepsilon(1 + |x|^2)^{\frac{n-1}{2}}$ . If  $\varepsilon$  is taken as indicated, we get  $|Q'(x)| \geq \frac{1}{2}|P'(x)|$ ; the result follows.

Let us now give a more general version of Theorem 1, valid if  $x$  has multiplicity  $k$ , empty if it has multiplicity  $k + 1$ :

**Theorem 4** *Let  $k \geq 1$  be an integer;  $P$  and  $Q$  be two polynomials of degree  $n$ , with  $|P - Q| \leq \varepsilon$ . If  $x$  is any zero of  $P$ , there exists a zero  $y$  of  $Q$ , with*

$$(15) \quad |x - y| \leq \left( \frac{n!}{(n-k)!} \frac{(1 + |x|^2)^{n/2}}{|Q^{(k)}(x)|} \right)^{1/k} \varepsilon^{1/k}.$$

If  $\varepsilon$  is small enough, namely

$$(16) \quad \varepsilon \leq \frac{(n-k)!}{2n!} \frac{|P^{(k)}(x)|}{(1 + |x|^2)^{\frac{n-k}{2}}}$$

then (15) implies

$$(17) \quad |x - y| \leq \left( \frac{2n!}{(n-k)!} \frac{(1 + |x|^2)^{n/2}}{|P^{(k)}(x)|} \right)^{1/k} \varepsilon^{1/k}.$$

**Proof of Theorem 4** It follows the same lines, so we only indicate the minor changes. We now set

$$(18) \quad f(\zeta) = [Q, (\bar{x}z + 1)^{n-k}(\bar{\zeta}z + 1)^k]$$

which is a polynomial in  $\zeta$  of degree  $k$ , satisfying

$$|f(x)| = |Q(x)| \leq \varepsilon'.$$

By Lemma 3, there is a point  $x'$ , with  $f(x') = 0$ , such that  $|x' - x| \leq (\varepsilon'/|a|)^{1/k}$ , where  $a$  is the coefficient of  $\zeta^k$  in (18), that is

$$a = [Q, (\bar{x}z + 1)^{n-k}z^k] = \frac{(n-k)!}{n!} Q^{(k)}(x).$$

So we get

$$(19) \quad |x' - x| \leq \left( \frac{n!}{(n-k)!} \frac{\varepsilon'}{|Q^{(k)}(x)|} \right)^{1/k}.$$

Let  $\varphi(u_1, \dots, u_n)$  be defined as before. We now get

$$\varphi(\underbrace{x, \dots, x}_{n-k \text{ times}}, \underbrace{x', \dots, x'}_{k \text{ times}}) = 0,$$

so by Walsh's principle, there is a point  $y$ , with  $\varphi(y, \dots, y) = 0$ , satisfying

$$|x - y| \leq \left( \frac{n!}{(n-k)!} \frac{\varepsilon'}{|Q^{(k)}(x)|} \right)^{1/k}.$$

This proves the first part of the Theorem. Now:

$$\begin{aligned} |P^{(k)}(x) - Q^{(k)}(x)| &= |[P^{(k)} - Q^{(k)}, (\bar{x}z + 1)^{n-k}]| \\ &= \frac{n!}{(n-k)!} |[P - Q, z^k(\bar{x}z + 1)^{n-k}]| \\ &\leq \frac{n!}{(n-k)!} \varepsilon (1 + |x|^2)^{\frac{n-k}{2}}, \end{aligned}$$

and the second part follows.

How sharp is the coefficient of  $\varepsilon$  in estimates (3) or (5)? We do not know exactly, but the order of magnitude is almost best possible. Indeed take  $P = z^n - 1$ , with  $x = 1$ , and  $Q = z^n + \varepsilon \sqrt{\binom{n}{n/2}} z^{n/2} - 1$  (for  $n$  even). Then  $[P - Q] = \varepsilon$ . The roots of  $Q$  are the  $n/2$  roots of

$$-\frac{\varepsilon}{2} \sqrt{\binom{n}{n/2}} \pm \sqrt{1 + \frac{\varepsilon^2}{4} \binom{n}{n/2}}$$

and if  $y$  is the real zero

$$\left( \sqrt{1 + \frac{\varepsilon^2}{4} \binom{n}{n/2}} - \frac{\varepsilon}{2} \sqrt{\binom{n}{n/2}} \right)^{2/n}.$$

We find

$$|x - y| \sim \frac{\varepsilon}{n} \sqrt{\binom{n}{n/2}} \sim \frac{\varepsilon}{n} 2^{n/2} \left( \frac{2}{\pi n} \right)^{1/4},$$

whereas estimates (3) gave  $2^{n/2} \varepsilon$ .

## 2 A Bound for the Largest Zero

We now give an estimate for the largest root of  $P$ , in terms of Bombieri's norm. This estimate may be substituted in the term  $1 + |x|^2$ , in Theorems 1 and 2 above. Of course, now, some normalization is necessary. We choose the usual one, that is  $a_n = 1$ .

**Theorem 5** *If  $P = \sum_0^n a_j z^j$  is a polynomial with  $a_n = 1$ , its roots  $x_1, \dots, x_n$  satisfy the estimate*

$$(20) \quad \max_j |x_j| \leq \sqrt{n[P]^2 - 1}.$$

*This estimate is best possible.*

**Proof** Let us order the roots so that  $|x_1| \geq |x_2| \geq \dots \geq |x_n|$ .

Applying Bombieri's inequality (see [2]) to the pair  $z - x_1, (z - x_2) \cdots (z - x_n)$  yields:

$$\begin{aligned} [P] &\geq \sqrt{\frac{1!(n-1)!}{n!}} [z - x_1] [(z - x_2) \cdots (z - x_n)] \\ &\geq \frac{1}{\sqrt{n}} (1 + |x_1|^2)^{1/2}, \end{aligned}$$

which gives (20).

The estimate (20) is best possible in the sense that, for every  $n$  and every  $\varepsilon > 0$ , there is a polynomial  $P$  which has a root satisfying

$$(21) \quad |x| \geq (1 - \varepsilon) \sqrt{n[P]^2 - 1}.$$

Indeed, with  $x > 0$ , consider  $P = (z - x)(z + \frac{1}{x})^{n-1}$ . Since the pair  $z - x, (z + \frac{1}{x})^{n-1}$  is extremal for the product (see Beauzamy [2]), we get

$$[P] = \frac{1}{\sqrt{n}} [z - x] \left[ z + \frac{1}{x} \right]^{n-1} = \frac{1}{\sqrt{n}} (1 + x^2)^{1/2} \left( 1 + \frac{1}{x^2} \right)^{\frac{n-1}{2}},$$

so

$$n[P]^2 - 1 = (1 + x^2) \left( 1 + \frac{1}{x^2} \right)^{n-1} - 1,$$

and the inequality

$$x^2 \geq (1 - \varepsilon)^2 \left( (1 + x^2) \left( 1 + \frac{1}{x^2} \right)^{n-1} - 1 \right),$$

is satisfied, for fixed  $n$  and  $\varepsilon$ , if  $x$  is large enough.

### 3 Blowing Up a Multiple Zero

Theorem 4 indicates that, if you start with a multiple zero  $x$  of  $P$ , of order  $k$ , and if you move  $P$  to  $Q$  with  $[P - Q] \leq \varepsilon$ , then  $x$  will be moved into  $y$ , with  $|x - y| \leq C\varepsilon^{1/k}$ . But when is such an estimate obtained? Are there cases where a better one holds? The answer is: if the multiple zero stays multiple, stronger estimates can be obtained; the worst case comes if the multiple zero "blows up" into single ones. We will describe this phenomenon in detail in the case of  $P = (z - a)^n$ .

- Case 1:  $Q$  has itself a multiple zero of order  $n$ ,  $Q = (z - b)^n$ . Then the condition  $[P - Q] \leq \varepsilon$  implies  $|b - a| \leq \varepsilon$ .

This is clear, from the formula  $[P']_{(n-1)} \leq n[P]_{(n)}$ , which itself is obtained by elementary manipulations of the binomial coefficients. Here we indicate by a suffix  $(n)$  or  $(n-1)$  which norm is used, so as to avoid any confusion.

- Case 2: all roots of  $Q$  are simple (or we have no information on  $Q$ ). Then (17), with  $k = n$ , gives for  $Q = (z - b_1) \cdots (z - b_n)$ :

$$(22) \quad |b_j - a| \leq 2^{1/n} (1 + |a|^2)^{1/2} \varepsilon^{1/n}.$$

This estimate is best possible in general: if  $Q = (z - a)^n - \varepsilon$ , then  $[P - Q] = \varepsilon$ , and  $|b_j - a| = \varepsilon^{1/n}$  for all  $j$ .

- Case 3: mixed case  $Q = (z - b)^k(z - b_1) \cdots (z - b_{n-k})$ . Then, first, the estimate  $|b - a| \leq \varepsilon^{1/n}$  can be improved, and we get

$$(23) \quad |b - a| \leq \varepsilon^{1/n-k+1} 2^{1/n-k+1} (1 + |a|^2)^{1/2}.$$

Indeed, we consider  $P^{(k-1)}$  and  $Q^{(k-1)}$  (which both have  $a$  and  $b$  respectively as zeros) and apply (22).

Then, also, we can obtain an estimate of the same form for  $b_1, \dots, b_{n-k}$ , namely

$$(24) \quad |b_j - a| \leq C(a, n) \varepsilon^{1/n-k+1}, \quad j = 1, \dots, n - k.$$

In order to prove (24), we first assume  $a = 0$ , that is

$$(25) \quad [z^n - (z - b)^k(z - b_1) \cdots (z - b_{n-k})] \leq \varepsilon,$$

and we know by (23) that

$$(26) \quad |b| = 0(\varepsilon^{1/n-k+1}).$$

We write  $\varepsilon' = \varepsilon^{1/n-k+1}$ . Let's also write

$$\begin{aligned} z^n - (z - b)^k(z - b_1) \cdots (z - b_{n-k}) &= c_1 z^{n-1} + c_2 z^{n-2} + \cdots + c_n \\ (z - b_1) \cdots (z - b_{n-k}) &= c'_1 z^{n-k} + c'_2 z^{n-k-1} + \cdots + c'_{n-k}. \end{aligned}$$

Then:

$$|c_1| = |kb + b_1 + \cdots + b_{n-k}| \leq \sqrt{\binom{n}{1}} \varepsilon.$$

Also, we have:

$$\begin{aligned} |c_{j+1}| &= \left| \binom{k}{j+1} b^{j+1} + \binom{k}{j} b^j c'_1 + \cdots + \binom{k}{l} b^l c'_{j-l+1} + \cdots + \binom{k}{1} b c'_j + c'_{j+1} \right| \\ &\leq \sqrt{\binom{n}{j+1}} \varepsilon. \end{aligned}$$

If we assume  $|c'_l| = 0(\varepsilon'^l)$ ,  $l = 1, \dots, j$ , we deduce from this formula that  $|c'_{j+1}| = 0(\varepsilon'^{j+1})$ , and so we have shown by induction that

$$(27) \quad |c'_j| = 0(\varepsilon'^j), \quad j = 1, \dots, n - k.$$

We need a lemma.

**Lemma 6** Let  $R = z^m + a_{m-1}z^{m-1} + \cdots + a_0$  be a polynomial where the coefficients  $a_{m-1}, \dots, a_0$  depend on some parameter  $\alpha$  and satisfy

$$|a_{m-1}| = 0(\alpha), |a_{m-1}| = 0(\alpha^2), \dots, |a_{m-k}| = 0(\alpha^k), \dots, a_0 = 0(\alpha^m),$$

when  $\alpha \rightarrow 0$ . Then all zeros of  $R$  are  $0(\alpha)$ ,  $\alpha \rightarrow 0$ .

This lemma is well-known and follows from estimates found for instance in Marden [5]. Let's give a quick proof. We have  $|a_{m-k}| \leq C\alpha^k$ ,  $k = 1, \dots, m$ . Let  $z$  be a zero of  $R$ . Then:

$$1 = -\frac{a_{m-1}}{z} \dots - \frac{a_k}{z^{m-k}} \dots - \frac{a_0}{z^m},$$

and so

$$1 \leq C \sum_1^\infty \left( \frac{|\alpha|}{|z|} \right)^k,$$

which implies  $|z| \leq (1 + C)|\alpha|$ . So the lemma is proved, and (24) follows from (27).

Let us now consider the general case,  $a \neq 0$ .

We define  $\tau_a P = P(z - a)$ . Our estimate will follow from the estimate in the case  $a = 0$  and the following.

**Lemma 7** For all  $P, Q$ , of degree  $n$ ,

$$[\tau_a P - \tau_a Q] \leq C(a, n)[P - Q],$$

where

$$C(a, n) = \max_{0 \leq l \leq n} \left\{ \binom{n}{l} (1 + |a|^2)^l \right\}^{1/2}.$$

**Proof of Lemma 7** We have

$$\begin{aligned} [\tau_a P]^2 &= \sum_{k=0}^n \frac{1}{\binom{n}{k} k!^2} \left| \sum_{j=0}^{n-k} P^{(k+j)}(0) \frac{a^j}{j!} \right|^2 \\ &\leq \sum_{k=0}^n \sum_{j=0}^{n-k} \frac{(n-k)! |a|^{2j}}{n! k! j!^2} |P^{(k+j)}(0)|^2 \\ &= \sum_{l=0}^n \sum_{j=0}^l \frac{(n-l+j)! l! |a|^{2j}}{(l-j)! j!^2 (n-l)!} \frac{|P^{(l)}(0)|^2}{l^2 \binom{n}{l}} \\ &\leq \left( \max_{0 \leq l \leq n} \sum_{j=0}^l \frac{(n-l+j)! l! |a|^{2j}}{(l-j)! j!^2 (n-l)!} \right) [P]^2. \end{aligned}$$

But

$$\begin{aligned} \sum_{j=0}^l \frac{(n-l+j)! l!}{(l-j)! j!^2 (n-l)!} |a|^{2j} &= \sum_{j=0}^l \binom{n-l+j}{j} \binom{l}{j} |a|^{2j} \\ &\leq \binom{n}{l} \sum_{j=0}^l \binom{l}{j} |a|^{2j} \\ &= \binom{n}{l} (1 + |a|^2)^l, \end{aligned}$$

and the lemma follows.

**Remark** We do not think that the above constant  $C(a, n)$  is sharp. One might think that  $(1 + |a|^2)^{n/2}$  is the right constant.

So we see that, starting with  $P = (z - a)^n$  and moving it to  $Q$  with  $[P - Q] \leq \varepsilon$ , the estimate  $|x - y| \leq \varepsilon^{1/n}$  can always be improved if one of the zeros of  $Q$  is multiple. The only case where it is sharp is the case where the multiple zero of  $P$  has blown up into  $n$  distinct simple zeros for  $Q$ .

As we already mentioned in [1], the combination of Bombieri's scalar product and Walsh Contraction Principle provides very efficient tools for the study of quantitative properties of polynomials: the proofs are simpler than the existing ones and the results are sharper. Other results on these lines will be published elsewhere.

## References

- [1] B. Beuzamy, *Finding the roots of polynomial equations: an algorithm with linear command*. To appear.
- [2] ———, *Product of many-variable polynomials: pairs that are maximal in Bombieri's norm*. J. Number Theory (1) **55**(1995), 129–143.
- [3] B. Beuzamy, E. Bombieri, P. Enflo, and H. Montgomery, *Products of polynomials in many variables*. J. Number Theory (2) **36**(1990), 219–245.
- [4] B. Beuzamy and J. Dégot, *Differential Identities*. Trans. Amer. Math. Soc. (7) **347**(1995), 2607–2619.
- [5] M. Marden, *Geometry of polynomials*. Mathematical Surveys, A.M.S., 1966.
- [6] A. Ostrowski, *Recherches sur la méthode de Graeffe*. Acta Math. **72**, 1940.
- [7] ———, *Solutions of equations and systems of equations (Appendix B)*. Academic Press, New-York, 1960.
- [8] Bruce Reznick, *An inequality for products of polynomials*. Proc. Amer. Math. Soc. (4) **117**(1993), 1063–1073.
- [9] J. L. Walsh, *On the location of the roots of certain type of polynomials*. Trans. Amer. Math. Soc. **12**(1922) 163–180.

*Société de Calcul Mathématique, S. A.*  
 111, Faubourg Saint Honoré  
 75008 Paris  
 France