THROUGH THE LOOKING GLASS

Book review by H.E.A. Campbell, Queen's University

Reflection Groups and Invariant Theory

Richard Kane
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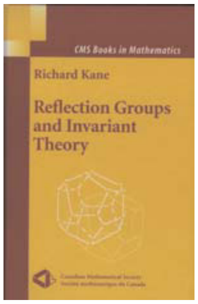

> *Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!*
>
> *Lewis Carroll, Through the Looking Glass*

In 1986, when I first started to think about the invariant theory of finite groups, there existed only one superb reference, the article by R.P. Stanley[1]. Since that time, there has been an explosion of interest, with many books and articles published, and conferences held. For example, there are two books by Bernd Sturmfels[2],[3], David Benson's book[4] *Polynomial Invariants of Finite Groups*, and Larry Smith's book[5] by the same title, as well as more recent books by Harm Derksen and Gregor Kemper[6] *Constructive Invariant Theory* and a book by Mara D Neusel and Larry Smith[7] *Invariant Theory of Finite Groups*. The book under review is most similar to the book by Humphrey[8], which covers some of the same material but is aimed at applications to Lie Algebras.

These books address quite different situations than the book under review: more general sorts of groups, consequently exhibiting less structure, and with different foci. Mathematicians seem to be interested in either Coxeter and Weyl groups, focusing on root systems and their combinatorics, or in *pseudo-reflection* groups and invariant theory. The former group are interested in Lie algebras and algebraic groups while the latter group includes commutative algebraists and topologists. This is the first book I've read that is aimed at both audiences, providing a clear exposition of a truly beautiful area of mathematics. It is probably fair to say that there isn't much new in the book, except to find all the relevant material in the one place, well-organized and well-explained. And so we have the advantage over Alice!

Let us accept that manifolds, their tangent bundles and the associated structure groups G — which are often the classical Lie groups (such as U_n) — embody very important aspects of physical systems, and therefore their study and properties are central to mathematics. It's one of the great triumphs of the mathematics of the 20th century that such bundles are classified by (homotopy classes) of maps of the base manifold into the "classifying" space BG of the structure group itself.

Mathematicians have expended considerable energy and enjoyed considerable success in examining the various algebraic and topological aspects of these classifying spaces. Indeed, it is fair to say that much of modern homotopy theory and algebraic topology has roots in this topic.

One of the most beautiful theorems I've ever encountered is the theorem of Borel. Suppose G is a compact Lie group with maximal torus T of rank, let's say, n. Let $W_G(T)$ be the Weyl group of G, that is, the normalizer of the torus T in G modulo the centralizer of the torus T in G. For example, suppose G is U_n the unitary group over the complex numbers. Then the Weyl group of the maximal torus is the symmetric group on n letters. Now the rational cohomology of the classifying space of the torus is a polynomial algebra over the rationals, and moreover, the Weyl group $W_G(T)$ can be shown to act as degree-preserving automorphisms of this algebra. Borel's theorem calculates the rational cohomology of the classifying space BG as the ring of invariants of the Weyl group. In the classical notation, the theorem is that

$$H^*(BG;Q) \cong Q[x_1, \ldots, x_n]^{W_G(T)}$$

For example, then

$$\mathrm{H}^*(BU_n;Q) \cong = Q[x_1, \ldots, x_n]^{\Sigma_\mathrm{n}} = Q[e_1, \ldots, e_n],$$

where e_i is the i-th elementary symmetric function in the x's.

The latter result, that functions invariant under all permutations of their variables can be written as polynomials in the elementary symmetric functions dates back to the time of Newton and Vandermonde. In part, the motivation at that time was to understand which univariate polynomials could be factored in terms of roots, the famous problem which led to Galois theory. If a univariate polynomial $p(t)$ of degree n does have n roots, say $x_1, \ldots, x_n$, we may write

$$p(t) = (t\text{-}x_1)(t\text{-}x_2) \cdots (t\text{-}x_n).$$

When we expand this product we obtain the elementary symmetric functions (up to sign) as the coefficients of the powers of t, and we observe that these are invariant under all permutations. This leads us to consider abstractly which polynomials in the x's are invariant under all permutations of the variables, and the classical result cited above.

What is most interesting about this result in the context of Kane's book is that the ring of invariants is again a polynomial algebra. Parts one and two of Kane's book could be summarized as the answer to the question: when does a finite group acting on a polynomial algebra have a polynomial algebra as its ring of invariants?

A *reflection* in Euclidean space is a linear map of the space which fixes (pointwise) a hyperplane and sends vectors orthogonal to the hyperplane to their negatives. And a *reflection group* is any group of transformations generated by such reflections. A modest generalization is a *pseudo-reflection* which is a linear map of finite order that fixes (pointwise) a hyperplane. In characteristic 0, where we have available an inner product, the condition of finite order implies that the vectors orthogonal to the hyperplane are mapped to a root of unity times themselves by a *pseudo-reflection*. However, in positive characteristic, a *pseudo-reflection* need not be diagonalizable.

Reflection groups and their generalizations *pseudo-reflection* groups have a lot of structure. In particular, the rich geometric and algebraic structure of the group is reflected by the rich algebraic structure of the associated ring of invariants.

The book has three parts in which these ideas are explored.

In the first part, reflection groups are studied in detail. For example, they may be classified by means of root systems, which provide the necessary combinatorial and algebraic data. Essentially, the geometric configuration of the reflecting hyperplanes (those associated to a set of generating reflections of the group) can be summarized in the concept of a root system, which leads naturally to the graphs and diagrams of Coxeter and Dynkin. This leads to other interesting associated ideas, such as the length of an element in a reflection group, which is, roughly speaking, the minimal number of reflections required to describe the element. Notably, the book provides a clear and understandable proof of an old theorem of Borel and de Siebenthal[9] describing maximal closed sub-root systems (page 136).

In the second part of the book, the invariant theory of these sorts of groups is studied. There is a thorough treatment of the beautiful result that in characteristic 0 a ring of invariants is a polynomial algebra if and only if the group is generated by *pseudo-reflections*. There is also a discussion of the result that, in arbitrary characteristic, if a group has a polynomial ring of invariants then the group must be generated by *pseudo-reflections*. This latter discussion is nearly self-contained, except for appeal to the Purity of Branch Loci theorem and Nakayama's lemma. These theorems have enormous appeal, linking the geometry of the representation of the group to the algebraic properties of the invariant ring.

This material also leads to work yet to be done. A modular group is a group whose order is divisible by the characteristic of the field. And, for example, there is as yet no characterization of modular groups with polynomial rings of invariants.

This part of the book goes on to present a number of fascinating auxiliary concepts: skew-invariants, extended rings of invariants, and more. One of the most interesting sections deals with harmonic polynomials and their relation to the ring of covariants. This latter ring is obtained as the quotient of the ring of all polynomials by the ideal generated by the positive degree invariants. Many of the arguments here are due to Steinberg, and there is a lovely characterization of reflection groups in terms of harmonic polynomials.

The third part of the book extends the first two parts, exploiting the interplay between the geometry and combinatorics of the groups and the algebraic structure of the invariant rings. Information is obtained about the conjugacy classes in reflection groups, regular elements are defined, their eigenvalues analyzed, and the natural description of the invariant ring as the coordinate ring of the orbit space associated to the group introduced. Further, a start is made on exploiting this relationship to understand conjugacy classes in *pseudo-reflection* groups. I believe this to be the first time this material has been gathered in one place. Moreover, there is still work to be done here too.

This is a very lovely book to read. I found the pace of the book to be perfect. The approach follows the development of the subject in the literature closely - this is how it was discovered, more or less. It appears to my eye to be accessible to a first-year graduate student. It is not overly terse, nor overly wordy and there are many examples.

References

[1] R. P. Stanley, Invariants of Finite Groups and their Applications to Combinatorics, Bull. AMS 1 (1979), No. 3, 475-511.

[2] B. Sturmfels, Algorithms in Invariant Theory, Springer, Berlin, 1993.

[3] B. Sturmfels, Gröbner Bases and Convex Polytopes, American Mathematical Society, 1996.

[4] D. J. Benson, Polynomial Invariants of Finite Groups, Lecture Note Series, vol. 190, Cambridge Univ. Press, London, 1993.

[5] L. Smith, Polynomial Invariants of Finite Groups, A K Peters, 1995.

[6] H. Derksen and G Kemper, Computational Invariant Theory, Encyclopaedia of Mathematical Sciences, Invariant Theory and Algebraic Transformation Groups, Vol. I, Springer, Berlin, 2002.

[7] M. Neusel and L Smith, Invariant Theory of Finite Groups, Mathematical Surveys and Monographs, vol. 94, AMS, 2002.

[8] J. E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge Studies in Advanced Mathematics, Vol. 29, Cambridge Univ. Press, 1992.

[9] A. Borel and J. de Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clo*s, Comment. Math. Helv. 23 (1949), 251-282.