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Simultaneous approximation and interpolation of increasing functions by increasing entire functions

When A and B are countable dense subsets of $\mathbb{R}$, it follows from a well-known result of Cantor that $f[A] = B$ for some order-isomorphism f of $\mathbb{R}$. A theorem of K. F. Barth and W. J. Schneider states that f can be taken to be the restriction to $\mathbb{R}$ of an entire function. S. Shelah established a consistent analog of Cantor's result for sets of cardinality $\aleph_1$ by building a model where $2^{\aleph_0} > \aleph_1$ and second category sets of cardinality $\aleph_1$ exist while any two sets of cardinality $\aleph_1$ which are nonmeager in every interval are order-isomorphic. In earlier work, we proved that the order-isomorphism in Shelah's theorem can be taken to be the restriction to $\mathbb{R}$ of an entire function. Using an approximation theorem of L. Hoischen, we also showed that the order-isomorphism f can be taken so that it and its first n derivatives approximate those of a given nondecreasing surjection g of class C^n. Hoischen's theorem also gives equality of the derivatives of f and g on a closed discrete set. We incorporate that improvement into our earlier result.

The following special case of the theorem is provable in ZFC. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^n surjection. Let $\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be positive and continuous. Let $E \subseteq \mathbb{R}$ be a closed discrete set on which g is strictly increasing. Let each of $\{A_i\}$, $\{B_i\}$ be a sequence of pairwise disjoint countable dense subsets of $\mathbb{R}$ such that for each $i \in \mathbb{N}$ and $x \in E$ we have $x \in A_i$ if and only if $g(x) \in B_i$. Then there is an entire function $f\colon \mathbb{C} \to \mathbb{C}$ such that $f[\mathbb{R}] \subseteq \mathbb{R}$ and the following properties hold.

(a) For all $x \in \mathbb{R} \setminus E$, $Df(x) > 0$.

(b) For $k = 0, \dots, n$ and all $x \in \mathbb{R}$, $|D^k f(x) - D^k g(x)| < \varepsilon(x)$.

(c) For $k = 0, \dots, n$ and all $x \in E$, $D^k f(x) = D^k g(x)$.

(d) For each $i \in \mathbb{N}$, $f[A_i] = B_i$.